Indian Statistical Institute

Midterm Examination 2024-2025

Analysis of Several Variables, B.Math Second Year

Time : 3 Hours Date : 09.09.2024 Maximum Marks : 100 Instructor : Jaydeb Sarkar

---

(1) (15 marks) Let $w = f(u, v)$, where

$$u = \frac{y - x}{xy} \text{ and } v = \frac{z - y}{yz},$$

for all $x, y, z \neq 0$. Compute

$$x^2 \frac{\partial w}{\partial x} + y^2 \frac{\partial w}{\partial y} + z^2 \frac{\partial w}{\partial z}.$$

(2) (15 marks) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Indicate whether the following identity is right or wrong, and justify your answer:

$$f_{xy}(0, 0) = f_{yx}(0, 0).$$

(3) (15 marks) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. If $f(0) = 0$, then prove that there exist functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$, such that

$$f(x) = \sum_{i=1}^{n} x_i g_i(x).$$

(4) (15 marks) Consider the function $F : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$F(x, y, z) = x + yz^2 + e^z.$$

Prove that there exists a differentiable function f defined in a neighborhood of $(-1, 1)$ such that

$$f(-1, 1) = 0,$$

and

$$F(x, y, f(x, y)) = 0.$$

Also, compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(-1, 1)$.

(5) (15 marks) Suppose that the equation $F(x, y, z) = 0$ can be solved for each of the variables as a differentiable function of the other two. Indicate whether the following identity is right or wrong, and justify your answer:

$$\frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} = 1.$$

(6) (15 marks) Let $f : \mathcal{O}_n \to \mathbb{R}^n$ be a C^1-function. Assume that $f'(x)$ is invertible for all $x \in \mathcal{O}_n$. Prove that f is an open map.

(7) (20 marks) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonlinear homogeneous function of degree one. Assume that

$$f(0) = 0.$$

Prove that f is not differentiable.